NIIG-DP-98-3
December, 1998
hep-lat-9812002

Ginsparg-Wilson Relation and Lattice Supersymmetry

Hiroto So* and Naoya Ukita†

Department of Physics,
Niigata University,
Ikarashi 2-8050, Niigata 950-2181, Japan.

Abstract

The Ginsparg-Wilson(G-W) relation is extended for supersymmetric free theories on a lattice. Exact lattice supersymmetry(SUSY) can be defined without any ambiguities in difference operators. The lattice action constructed by a block-spin transformation is invariant under the symmetry. $U(1)_R$ symmetry on the lattice is also realized as one of exact symmetries. For an application, the extended G-W relation is given for a two-dimensional model with chiral-multiplets. It is argued that the relation may be generalized for interacting cases.

*so@muse.hep.sc.niigata-u.ac.jp
†ukita@muse.hep.sc.niigata-u.ac.jp

1. *Introduction* Recently, there have been much progress in understanding chiral symmetry on a lattice. Lüscher has given a lattice form of a chiral transformation, which differs from the one in the continuum theory by a term proportional to lattice spacing[1]. This result may contain an important step to escape the no-go theorem[2]. The lattice fermion action is invariant under the lattice chiral symmetry if the Dirac operator satisfies the Ginsparg-Wilson(G-W) relation[3]. The relation should be recognized as a remnant of the continuum chiral symmetry and a kind of Ward-Takahashi identity. A particular solution has been given by Neuberger[4].

In this paper, we aim at the construction of an exact lattice supersymmetry(SUSY) consistent with the SUSY-extension of the G-W relation. The first step to get the exact SUSY in the spirit of Lüscher is to restate a continuum SUSY as a naïve lattice SUSY without any ambiguities in difference operators. To this end, we must introduce a block-spin function which transforms field variables in the continuous space into dynamical ones on the lattice. The second step is to define an exact SUSY transformation from the naïve lattice version, where we use our SUSY-extension of the G-W relation. Our final step is to determine an arbitrary parameter in the definition of the exact SUSY transformation. It should be consistent with the lattice algebra. Furthermore, we find a $U(1)_R$ charge on the lattice in the spirit of Lüscher. There appears no arbitrary parameter in the charge. Finally, we find exact SUSY and $U(1)_R$ invariance of lattice theories without doubling problems.

In order to see how our approach works, we construct a supersymmetric free theory in two dimensions. In addition to fermionic and bosonic G-W relations used by Kikukawa-Aoyama[5], we can derive the more relations for the lattice action. Once one of kinetic terms for component fields is found, we can construct a SUSY-invariant total action.

Our approach can be generalized for interacting cases, owing to the SUSY-extension of the G-W relation. Unlike previous works on lattice SUSY[5-7], we need not to introduce the definition of difference operators by hand. Therefore, the problem of Leibniz rule may be overcome.

2. *Derivation of SUSY Ginsparg-Wilson relation* The original G-W relation was derived as an identity for the Gaussian type effective action. The relation may play an important role in characterizing a lattice analog of the chiral symmetry which should be realized in the vicinity of the continuum limit.

A way out of the no-go theorem is to introduce two chiral matter fields. A SUSY transformations for two chiral-multiplets $\Phi_j = (\phi_j, \psi_j, F_j)^{\mathrm{T}}$ $j = 1, 2$ in the continuum theory are defined as

$$\delta_\epsilon \Phi_j = Q(\epsilon, \bar{\epsilon})\Phi_j, \tag{1}$$

$$\delta_\epsilon \bar{\Phi}_j = \bar{\Phi}_j \bar{Q}(\epsilon, \bar{\epsilon}), \tag{2}$$

where T represents a transpose operation and the space-time is assumed to be Euclidean. In the rest of the section, we suppress the index j for chiral-multiplets. Starting with a continuum theory, we define its regularized theory on a cubic lattice by performing a block-spin transformation. A lattice point is expressed by an integer vector $\{n_\mu a\}$, where a is lattice constant. We take $a = 1$ for simplicity. The block-spin transformation from $\Phi(x)$ to Φ_n is given by

$$\Phi_n \sim \int \mathrm{d}x f_n(x)\Phi(x) \quad \equiv \quad \langle f_n, \Phi\rangle, \tag{3}$$

$$\bar{\Phi}_n \sim \int \mathrm{d}x f_n(x)\bar{\Phi}(x) \quad \equiv \quad \langle f_n, \bar{\Phi}\rangle, \tag{4}$$

where $f_n(x) = f(x - n)$ is a block-spin function with finite support around $x_\mu = n_\mu$. $\langle , \rangle$ implies the usual inner product in a function space.

Following to Ginsparg and Wilson[3], we may define a Gaussian effective action A_{eff} by using a SUSY-invariant massless action A_c in the continuum theory*:

$$\begin{aligned} &\exp(-A_{\mathrm{eff}}[\Psi_n, \bar{\Psi}_n]) \\ = \int \mathcal{D}\Phi(x)\mathcal{D}\bar{\Phi}(x) &\exp(-\sum_{n,m}(\bar{\Psi}_n - \bar{\Phi}_n)\boldsymbol{\alpha}_{n,m}(\Psi_m - \Phi_m) - A_c[\Phi, \bar{\Phi}]). \end{aligned} \tag{5}$$

Here $\alpha_{n,m}$ is a matrix acting on the multiplet Ψ_n,

*Since we have two chiral multiplets, it is possible to construct a Dirac mass term.

$$\boldsymbol{\alpha}_{n,m} = \alpha\ \delta_{n,m} \begin{pmatrix} 0 & 0 & 1 \\ 0 & V & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{6}$$

where V is some anti-symmetric matrix determined by a mass-term in a SUSY-invariant Lagrangian and α is proportional to $O(a^{-1})$.

We may define naïve lattice SUSY by restating the continuum SUSY as follows:

$$\begin{aligned} \delta_\epsilon^N \Phi_n &= \int f_n(x)\delta\Phi(x)\mathrm{d}x \\ &= Q_L(\overrightarrow{\bigtriangledown})\Phi_n, \end{aligned} \tag{7}$$

$$\begin{aligned} \delta_\epsilon^N \bar{\Phi}_n &= \int f_n(x)\delta\bar{\Phi}(x)\mathrm{d}x \\ &= \bar{\Phi}_n \bar{Q}_L(\overleftarrow{\bigtriangledown}). \end{aligned} \tag{8}$$

A derivative operator in the continuum SUSY is replaced by a difference operator in the lattice SUSY using the relation $\partial_\mu f_n(x) = -\overrightarrow{\bigtriangledown}_\mu f_n(x)$. Although the explicit form of the difference operator depends on the block-spin function, it is possible to choose a reasonable function for the realization of lattice SUSY. As a preparation for the exact lattice SUSY, we would just look at some properties of the naïve SUSY.

Under this naïve transformation, our effective action changes by

$$\begin{aligned} &\exp(-A_{\text{eff}}[\Psi', \bar{\Psi}']) \\ = &\int \mathcal{D}\Phi(x)\mathcal{D}\bar{\Phi}(x) \exp(-(\bar{\Psi}' - \bar{\Phi})\boldsymbol{\alpha}(\Psi' - \Phi) - A_c[\Phi, \bar{\Phi}]) \\ = &\int \mathcal{D}\Phi(x)\mathcal{D}\bar{\Phi}(x) \exp(-(\bar{\Psi} - \bar{\Phi}')e^{\bar{Q}_L}\boldsymbol{\alpha}e^{Q_L}(\Psi - \Phi') - A_c[\Phi', \bar{\Phi}']), \end{aligned} \tag{9}$$

where the lattice site and the spinor indices are both omitted. It is assumed that the A_c is invariant under SUSY transformation in the continuum theory,

$$A_c[\Phi,\bar{\Phi}] = A_c[\Phi',\bar{\Phi}']. \tag{10}$$

Although the path-integral measure is naïvely unchanged

$$\mathcal{D}\Phi'\mathcal{D}\bar{\Phi}' = \mathcal{D}\Phi\mathcal{D}\bar{\Phi}, \tag{11}$$

we take account of the contribution of the Jacobian factor which is needed to consider a possible effect of the anomaly[8].

Let us derive a SUSY extension of the G-W relation for a free theory described by

$$A_{\rm eff}[\Psi,\bar{\Psi}] = \sum_{n,m} \bar{\Psi}_n S_{(n,m)} \Psi_m. \tag{12}$$

Under the naïve SUSY , it transforms as

$$\begin{aligned} &\exp(-A_{\rm eff}[\Psi,\bar{\Psi}])(1-(\bar{\Psi}(SQ_L+\bar{Q}_L S)\Psi) \\ = \;& (1+\delta J - \mathrm{str}\ \boldsymbol{\alpha}^{-1}(\boldsymbol{\alpha}Q_L+\bar{Q}_L\boldsymbol{\alpha})\boldsymbol{\alpha}^{-1}S + \mathrm{str}\ \boldsymbol{\alpha}^{-1}(\boldsymbol{\alpha}Q_L+\bar{Q}_L\boldsymbol{\alpha}) \\ & -\bar{\Psi}S\boldsymbol{\alpha}^{-1}(\boldsymbol{\alpha}Q_L+\bar{Q}_L\boldsymbol{\alpha})\boldsymbol{\alpha}^{-1}S\Psi)\exp(-A_{\rm eff}[\Psi,\bar{\Psi}]), \end{aligned} \tag{13}$$

where δJ comes from a Jacobian factor. So, we can get following two relations:

$$\delta J = \mathrm{str}\ \boldsymbol{\alpha}^{-1}(\boldsymbol{\alpha}Q_L(\overrightarrow{\bigtriangledown})+\bar{Q}_L(\overleftarrow{\bigtriangledown})\boldsymbol{\alpha})\boldsymbol{\alpha}^{-1}S - \mathrm{str}\ \boldsymbol{\alpha}^{-1}(\boldsymbol{\alpha}Q_L(\overrightarrow{\bigtriangledown})+\bar{Q}_L(\overleftarrow{\bigtriangledown})\boldsymbol{\alpha}), \tag{14}$$

and

$$\bar{\Psi}(SQ_L(\overrightarrow{\bigtriangledown})+\bar{Q}_L(\overleftarrow{\bigtriangledown})S)\Psi = \bar{\Psi}S\boldsymbol{\alpha}^{-1}(\boldsymbol{\alpha}Q_L(\overrightarrow{\bigtriangledown})+\bar{Q}_L(\overleftarrow{\bigtriangledown})\boldsymbol{\alpha})\boldsymbol{\alpha}^{-1}S\Psi. \tag{15}$$

These are SUSY extended G-W relations. Note that the right hand sides of these relations vanish if the difference operator($\overrightarrow{\bigtriangledown}$) is anti-symmetric in the matrix notation.

For an exact lattice SUSY transformation, the above relations suggest us to define

$$q \equiv Q_L(\overrightarrow{\bigtriangledown}) - Q_L(\overrightarrow{\bigtriangledown})\boldsymbol{\alpha}^{-1}S \tag{16}$$

$$\bar{q} \equiv \bar{Q}_L(\overleftarrow{\bigtriangledown}) - S\boldsymbol{\alpha}^{-1}\bar{Q}_L(\overleftarrow{\bigtriangledown}) \tag{17}$$

under which we can show the invariance of our effective action:

$$\delta A_{\text{eff}} = \bar{\Psi}(Sq + \bar{q}S)\Psi = 0. \tag{18}$$

This is a SUSY extension of Lüscher's symmetry[1]. Similar to chiral symmetry, SUSY is also modified by $O(a)$ because of $\boldsymbol{\alpha} = O(a^{-1})$. We must note that there may exist an arbitrary parameter c in defining lattice SUSY:

$$q_c \equiv q - cQ_L(\overrightarrow{\bigtriangledown}_{\text{s}})\boldsymbol{\alpha}^{-1}S, \tag{19}$$

$$\bar{q}_c \equiv \bar{q} - cS\boldsymbol{\alpha}^{-1}\bar{Q}_L(\overleftarrow{\bigtriangledown}_{\text{s}}), \tag{20}$$

where the symmetric difference operator $\overrightarrow{\bigtriangledown}_{\text{s}} \equiv \dfrac{\overrightarrow{\bigtriangledown} - \overrightarrow{\bigtriangledown}^{\text{T}}}{2}$. The parameter should be determined by the closure property of the algebra. Since this transformation is not unitary, we get the Jacobian factor $1 - \delta J$ under the transformation Eqs.(19) and (20); δJ is expressed as Eq.(14) and is easily shown to be independent of the parameter c.

For $U(1)_R$ charge Q_R, we find the G-W relations similar to the SUSY case:

$$\delta J_R = \text{str } \boldsymbol{\alpha}^{-1}(\boldsymbol{\alpha}Q_R + \bar{Q}_R\boldsymbol{\alpha})\boldsymbol{\alpha}^{-1}S - \text{str } \boldsymbol{\alpha}^{-1}(\boldsymbol{\alpha}Q_R + \bar{Q}_R\boldsymbol{\alpha}), \tag{21}$$

and

$$\bar{\Psi}(SQ_R + \bar{Q}_RS)\Psi = \bar{\Psi}S\boldsymbol{\alpha}^{-1}(\boldsymbol{\alpha}Q_R + \bar{Q}_R\boldsymbol{\alpha})\boldsymbol{\alpha}^{-1}S\Psi. \tag{22}$$

The conserved '$U(1)_R$' charge [5],

$$q_R \equiv Q_R(1 - \boldsymbol{\alpha}^{-1}S), \tag{23}$$

can be also found and has no arbitrary parameter unlike the SUSY case.

3. *An example: 2-Dimensional chiral-multiplets* We consider two chiral-multiplets Φ_j, $j = 1, 2$, consisted of real scalars ϕ_j, auxiliary fields F_j and complex Weyl spinors χ_j. These are arranged in a complex multiplet $\Phi = (\phi_1 + i\phi_2, \chi_1 + i\chi_2, \chi_1^* + i\chi_2^*, F_1 + iF_2)^{\mathrm{T}} \equiv (\phi, \chi, \bar{\chi}, F)^{\mathrm{T}}$ and its conjugate $\bar{\Phi} = (\phi_1 - i\phi_2, \chi_1 - i\chi_2, \chi_1^* - i\chi_2^*, F_1 - iF_2)) \equiv (\phi^*, \bar{\chi}^\dagger, \chi^\dagger, F^*)$. We define N=1 SUSY transformation:

$$\left\{ \begin{array}{l} \delta_\epsilon \phi = i(\epsilon^* \chi + \epsilon \bar{\chi}) \\ \delta_\epsilon \chi = -2\epsilon^* \partial_z \phi + i\epsilon F \\ \delta_\epsilon \bar{\chi} = -2\epsilon \partial_{\bar{z}} \phi - i\epsilon^* F \\ \delta_\epsilon F = -2\epsilon \partial_{\bar{z}} \chi + 2\epsilon^* \partial_z \bar{\chi}. \end{array} \right. \tag{24}$$

We consider a SUSY-invariant massless Lagrangian:

$$\mathcal{L} = 2\partial_{\bar{z}} \phi^* \partial_z \phi + i(\chi^\dagger \partial_z \bar{\chi} + \bar{\chi}^\dagger \partial_{\bar{z}} \chi) - \frac{1}{2} F^* F. \tag{25}$$

Then, we obtain the matrix V in Eq.(6),

$$V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

from a mass-term in a SUSY-invariant Lagrangian,

$$\mathcal{L}_m = -\frac{m}{2}(F^* \phi + F \phi^* + \chi^\dagger \chi - \bar{\chi}^\dagger \bar{\chi}). \tag{26}$$

The naïve lattice SUSY takes of the matrix form:

$$Q_L = \begin{pmatrix} 0 & i\epsilon^* & i\epsilon & 0 \\ -2\epsilon^* \overrightarrow{\bigtriangledown}_z & 0 & 0 & i\epsilon \\ -2\epsilon \overrightarrow{\bigtriangledown}_{\bar{z}} & 0 & 0 & -i\epsilon^* \\ 0 & -2\epsilon \overrightarrow{\bigtriangledown}_{\bar{z}} & +2\epsilon^* \overrightarrow{\bigtriangledown}_z & 0 \end{pmatrix}, \tag{27}$$

and

$$\bar{Q}_L = \begin{pmatrix} 0 & -2\epsilon^* \overleftarrow{\bigtriangledown}_z & -2\epsilon \overleftarrow{\bigtriangledown}_{\bar{z}} & 0 \\ -i\epsilon^* & 0 & 0 & 2\epsilon \overleftarrow{\bigtriangledown}_{\bar{z}} \\ -i\epsilon & 0 & 0 & -2\epsilon^* \overleftarrow{\bigtriangledown}_z \\ 0 & i\epsilon & -i\epsilon^* & 0 \end{pmatrix}. \tag{28}$$

It follows that

$$\boldsymbol{\alpha} Q_L + \bar{Q}_L \boldsymbol{\alpha} = \alpha \begin{pmatrix} 0 & -2\epsilon (TD)_{\bar{z}} & 2\epsilon^* (TD)_z & 0 \\ 2\epsilon (TD)_{\bar{z}} & 0 & 0 & 0 \\ -2\epsilon^* (TD)_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{29}$$

where $(TD)_z$ denotes a total derivative $\overrightarrow{\bigtriangledown}_z + \overleftarrow{\bigtriangledown}_z$.

We obtain in our approach not only the original G-W relation but also the relation among kinetic terms for fermion, boson and auxiliary fields:

$$|S_{\bar{\chi}^\dagger,\chi}|^2 + |S_{\chi^\dagger,\chi}|^2 = -\alpha S_{\chi^\dagger,\chi}, \tag{30}$$

$$S_{F^*,F} S_{\phi^*,\phi} + |S_{F^*,\phi}|^2 = \alpha S_{F^*,\phi}, \tag{31}$$

$$2 S_{F^*,F} (TD)_{\bar{z}} S_{\chi^\dagger,\chi} = \alpha(-2 S_{F^*,F} \overrightarrow{\bigtriangledown}_{\bar{z}} + i S_{\bar{\chi}^\dagger,\chi}), \tag{32}$$

$$2 S_{\phi^*,F} (TD)_z S_{\bar{\chi}^\dagger,\chi} = i\alpha(S_{\phi^*,\phi} + 2i \overleftarrow{\bigtriangledown}_z S_{\bar{\chi}^\dagger,\chi}). \tag{33}$$

Once we find the fermion kinetic term, $S_{\bar{\chi}^\dagger,\chi}$, it is easy to express the total action explicitly from these relations.

4. *Conclusions* In this letter, we have extended the Ginsparg-Wilson relation for a supersymmetric case and applied it for free theories†. The exact lattice SUSY and $U(1)_R$ symmetry are found. As an example, SUSY-extended G-W relations of 2-dimensional two chiral-multiplets are derived.

†Bietenholz also studied SUSY-extended G-W relations by similar approach [9], but his expression for G-W relations is quite different from ours.

Similarly, for a Majorana fermion, one can describe a SUSY theory by single non-chiral matter field escaping from the no-go theorem.

Our block-spin construction uniquely fixes the definition of difference operators owing to our block-spin construction. If one uses the even block-spin function which leads us to $\boldsymbol{\alpha} Q_L + \bar{Q}_L \boldsymbol{\alpha} = 0$, our lattice SUSY algebra is exactly closed.

For interacting cases, we can also derive the G-W relation. If we use even block-spin functions, the naïve SUSY is an exact symmetry for the block-spin effective action and we may overcome the problem of Leibniz rule. Further investigation to the G-W relation is necessary to show these properties in concrete terms. Finally, this G-W relation is generally important for understanding the continuum limit with the supersymmetry. The detailed analysis shall be reported in a forthcoming paper[10].

We are grateful to Y. Igarashi for reading our manuscript carefully and invaluable comments. We also thank K. Itoh and H. Nakano for their encouragement.

References

[1] M. Lüscher, Phys. Lett. **B428** (1998) 342.

[2] H. B. Nielsen and M. Ninomiya, Nucl. Phys. **B185** (1981) 20; **B195** (1982) 541; **B193** (1981) 173.

[3] P. Ginsparg and K. Wilson, Phys. Rev. **D25** (1982) 2649.

[4] H. Neuberger, Phys. Lett. **B417** (1998) 141; **B427** (1998) 353.

[5] T. Aoyama and Y. Kikukawa, Phys. Rev. **D59** (1999) 054507.

[6] N. Sakai and M. Sakamoto, Nucl. Phys. **B229** (1983) 173.

[7] S. Nojiri, Prog. Theor. Phys. **74** (1985) 819.

[8] K. Fujikawa, Phys. Rev. **D21** (1980) 2848; **D22** (1980) 1499.

[9] W. Bietenholz, Mod. Phys. Lett. **A14** (1999) 51.

[10] M. Koseki, H. So and N. Ukita, in preparation.